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Solutions for Two-Dimensional Dilaton Gravity

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In a recent paper Frolov, Hendy, and Larsen (1996) obtained a set of two nonlinear second-order coupled partial differential equations of 2D dilaton gravity. They obtained a special solution assuming all the dependent variables were independent of time. In the present work we reduce the above set of equations to a simple form, from which we obtain a class of solutions that includes the solution of Frolov, Hendy, and Larsen as a special case.

In a recent paper Frolov *et al.*[(1)] showed that 2D string holes can be obtained as solutions of 2D dilaton gravity with a suitably chosen dilaton potential. For this they considered the following action of 2D dilaton gravity

$$S = \frac{1}{2\pi} \int dt\, dx \quad \sqrt{-q}\, e^{-2\phi}\, [R + 2\, (\nabla\phi)^2 + V(\phi)] \tag{1}$$

where the dilaton potential $V(\phi)$ is unspecified. Choosing the two-dimensional conformal gauge as

$$g_{\mu\nu} = e^{2\phi} \times \mathrm{diag}(-1, 1), \qquad \rho = \rho(t, x) \tag{2}$$

Frolov *et al.* reduced the action (1) to

$$S = \frac{1}{\pi} \int dt\, dx\, e^{-2\phi} \left[\rho_{tt} - \rho_{xx} + \phi_x^2 - \phi_t^2 + \frac{1}{2}\, e^{2\rho} + V(\phi) \right] \tag{3}$$

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They obtained that the field equations corresponding to the action (3) are

$$\rho_{xx} - \rho_{tt} + \phi_{tt} - \phi_{xx} + \phi_x^2 - \phi_t^2 + \frac{1}{4} e^{2\rho}(V' - 2V) = 0 \tag{4a}$$

$$\phi_{xx} - \phi_{tt} + 2(\phi_t^2 - \phi_x^2) + \frac{1}{2} e^{2\rho} V = 0 \tag{4b}$$

where $V' \equiv dV/d\phi$.

Considering both ρ and ϕ as functions of x only, Frolov *et al.* showed that

$$V(\phi) = \left[\frac{2}{r^2} (rF)_{,r} \right] \Big|_{r=e^{-\phi}/\lambda} \tag{5a}$$

$$\phi = -\ln|\lambda r|, \qquad \lambda = \text{const} \tag{5b}$$

together with an arbitrary function $F(r)$ are solutions of Eqs. (4), where

$$\frac{dr}{F(r)} = dx, \qquad e^{2\rho} = F \tag{6}$$

In the present work we extend the work of Frolov *et al.*[1] and solve these equations completely.

To this end, putting

$$U = 2\rho - \phi, \qquad Z = e^{-2\phi} \tag{7}$$

we can write Eqs. (4) as

$$u_{xx} - u_{tt} = \frac{e^u(2ZV_Z + V)}{2\sqrt{Z}}$$

$$Z_{xx} - Z_{tt} = e^u V \sqrt{Z}$$

which can be further rewritten as

$$Y_Z = 4U_{pq}/e^u \tag{8a}$$

$$y = 4Z_{pg}/e^u \tag{8b}$$

where

$$Y = V\sqrt{Z} \tag{8c}$$

and

$$x + t = p, \qquad x - t = q \tag{9}$$

From (8b)

$$Y_Z Z_p = \left(\frac{4Z_{pq}}{e^u}\right)_p$$

$$Y_Z Z_q = \left(\frac{4Z_{pq}}{e^u}\right)_q$$

Now we use Eq. (8a) with the above equations to get

$$\frac{U_{pq}}{e^u} = \frac{1}{Z_p}\left(\frac{Z_{pq}}{e^u}\right)_p, \qquad Z_p \neq 0$$

$$\frac{U_{pq}}{e^u} = \frac{1}{Z_q}\left(\frac{Z_{pq}}{e^u}\right)_q, \qquad Z_q \neq 0$$

which can be rewrittten as

$$(U_p\, Z_p)_q = (Z_{pp})_q, \qquad Z_p \neq 0 \tag{10a}$$

$$(U_q\, Z_q)_p = (Z_{qq})_p, \qquad Z_q \neq 0 \tag{10b}$$

which upon integration give

$$U_p Z_p = Z_{pp} + A(p), \qquad Z_p \neq 0 \tag{11a}$$

$$U_q Z_q = Z_{qq} + B(q), \qquad Z_q \neq 0 \tag{11b}$$

where $A(p)$ and $B(q)$ are functions of p and q, respectively.

Thus Eqs. (4) have been reduced to Eqs. (11) where ρ and ϕ can be obtained using (7), and from (8b) and (8c), V is given by

$$V = 4Z_{pq}/\sqrt{\bar{Z}} \tag{12}$$

Particular solutions of Eqs. (11) can be obtain with the ansatz

$$Z = Z(\psi) \tag{13a}$$

$$Y = \xi(p) + \eta(q) \tag{13b}$$

where $\xi(p)$ and $\eta(q)$ are arbitrary functions of p and q, respectively, and p, q are given by (4).

For our ansatz (13), Eqs. (11) takes the form

$$U_p = \frac{Z_{\psi\psi}\xi_p^2 + Z_\psi \xi_{pp} + A(p)}{Z_\psi \xi_p} \tag{14a}$$

$$U_q = \frac{Z_{\psi\psi}\eta_q^2 + Z_\psi \eta_{qq} + B(q)}{Z_\psi \eta_q} \tag{14b}$$

$$Z_\psi \xi_p \eta_q \neq 0 \tag{14c}$$

Using

$$\frac{\partial}{\partial p}\left(\frac{\partial u}{\partial q}\right) = \frac{\partial}{\partial q}\left(\frac{\partial u}{\partial p}\right)$$

from (14a) and (14b) we have

$$Z_{\psi\psi}(\xi_p^2 B(q) - \eta_q^2 A(q)) = 0 \tag{15}$$

From Eqs. (12) and (13) one sees that if $Z_{\psi\psi} = 0$, $d\xi/dp = 0$, or $d\eta/dq = 0$, then $V = 0$.

So, assuming $V(\phi) \neq 0$, one gets $Z_{\psi\psi} \neq 0$, $\xi_p \neq 0$, and $\eta_q \neq 0$.

Then Eq. (15) takes the form

$$A(p)/\xi_p^2 = B(q)/\eta_q^2 \tag{16}$$

Since the left-hand side of (16) is a function of p only and the right-hand side is a function of q only, both sides are equal to a constant, say M.

Then from (16) we have

$$A(p) = M\xi_p^2 \tag{17a}$$

$$B(q) = M\eta_q^2 \tag{17b}$$

Using (17a) and (17b) in Eqs. (14a) and (14b) and then integrating (14a) and (14b), we obtain

$$U = \ln\left|KZ_\psi\, \xi_p\eta_q\right| + M\int \frac{d\psi}{Z\psi} \tag{18}$$

where K is a constant of integration.

Using Eqs. (13), (15), and (18), we can obtain Eq. (12) as

$$V = 4Z_{\psi\psi}/K\, Z^{1/2} Z_\psi e^M \int \frac{d\psi}{Z\psi} \tag{19}$$

where Z is an arbitrary function of ψ.

In view of Eqs. (15) and (16) we have from (7) that

$$\phi = -\frac{1}{2}\ln\left|Z_\psi\right| \tag{20}$$

where Z is an arbitrary function of ψ, and ψ is given by Eq. (13b). Using (7) and (20), we can rewrite Eqs. (18) and (19) as

$$\rho = \frac{1}{2} \ln \left| \frac{-2K\phi_\psi \xi_p \eta_q}{e^\phi} \right| - \frac{M}{4} \int \frac{e^{2\phi}}{\phi_\psi} d\psi \tag{21}$$

$$v = \frac{4(\phi_{\psi\psi} - 2\phi_\psi^2)}{K\phi_\psi} \exp\left[\phi + \frac{M}{2} \int \frac{e^{2\phi}}{\phi_\psi} d\psi \right] \tag{22}$$

Thus the solutions of Eqs. (4) are given by Eqs. (20)–(22).

In particular, taking

$$\xi(p) = \xi(x + t) = \frac{x + t}{2} \tag{23a}$$

$$\eta(q) = \eta(x - t) = \frac{x - t}{2} \tag{23b}$$

we have from (13b) that

$$\psi = x \tag{24}$$

Using (23) and (24) and taking

$$M = 0$$

$$K = 2/\lambda$$

$$Z(\psi) = Z(x) = \lambda^2 r^2 \tag{25}$$

where λ = const, $r = r(x)$, and $r(x)$ is an arbitrary function of x, one can easily rewrite Eqs. (20)–(22) as

$$\phi = -\ln|\lambda r| \tag{26}$$

$$e^{2\rho} = r_x \tag{27}$$

$$V = \left[\frac{2}{r^2} (rF)_{,r} \right]_{|r=e^{-\phi}/\lambda} \tag{28}$$

where $F(r)$ is defined by

$$dr^*/F(r) = dx, \qquad e^{2\rho} = F \tag{29}$$

which are the particular solutions of Eqs. (4) obtained by Frolov *et al.*[1]

CONCLUSION

We have reduced the equations of 2D dilaton gravity (4a) and (4b) to a simple form given by Eqs. (11a), (11b), and (12). We have obtained a class of solutions of these equations with an ansatz given by (13a) and (13b). These solutions are given by equations (20)–(22) for nonzero $V(\phi)$. The solutions obtained by Frolov *et al.*[1] form a special case of the solutions given here.

REFERENCES

1. V. Frolov, S. Hendy, and A. L. Larsen, *Phys. Rev D* **54**, 5093 (1996).